Section 2.4 Schöberl's thesis

Gonzalo G. de Diego

July 6, 2020

1 Overview of section 2.4

In section 2.4 of his thesis, Schöberl proceeds to apply the theory developed in section 2.3 to three particular sets of equations. The application of the theory of section 2.3 consists in:

1. Construct the primal system

$$A^\varepsilon(u, v) = f(v)$$

   and the dual system

$$B^\varepsilon((u, p), (v, q)) = f(v).$$

2. Consider well-posedness of the system at $\varepsilon = 0$ in $V \times Q_0$ (Theorem 2.8). Remember that Q_0 is $\overline{\Lambda V}$ endowed with the norm:

$$||p||_{Q_0} \simeq \sup_{v \in V} \frac{c(p, \Lambda v)}{||v||}.$$

3. Define a norm $V \times Q$ and consider uniform well-posedness for $0 < \varepsilon \leq 1$ in $V \times Q$ (Theorem 2.9). The norm on Q is given by

$$||p||_Q^2 = ||p||_{Q_0}^2 + \varepsilon ||p||_c^2.$$

4. Choose elements for a non-conforming discretisation $V_h \times Q_h$ with bilinear form A_h^ε and prove stability of the discrete system.

2 Nearly incompressible materials

We analyse the equations from linear elasticity and consider the stability of the formulation in the incompressible limit. Note that the same primal formulation arises with Stokes when enforcing incompressibility with a penalty term [3] or with an augmented Lagrangian term [2].

2.1 Primal and dual formulations

The **primal** system for linear elasticity is set in

$$V = \left[H^1_{0,D}\right]^2$$

with the bilinear form

$$A^\varepsilon(u,v) = \underbrace{(e(u),e(v))_0}_{=:a(u,v)} + \varepsilon^{-1}\underbrace{(\operatorname{div} u, \operatorname{div} v)_0}_{=:c(\Lambda u,\Lambda v)}.$$

By defining $p = \varepsilon^{-1}\operatorname{div} u$, we get the **dual** formulation, defined in terms of the bilinear form

$$B^\varepsilon((u,p),(v,q)) = (e(u),e(v)_0 + (\operatorname{div} u, q)_0 + (\operatorname{div} v, p)_0 - \varepsilon(p,q)_0.$$

Thanks to Korn's inequality, which states that

$$||u||_1 \lesssim ||e(u)||_0 \quad \forall u \in V$$

we have the coercivity of $a(\cdot,\cdot)$.

2.2 The Q_0 and Q norms

2.2.1 Pure Dirichlet boundary conditions

If we have pure Dirichlet boundary conditions, i.e. $\Gamma_D = \partial\Omega$, then

$$||p||_{Q_0} := ||p||_{L^2/\mathbb{R}}$$

because

$$||p||_{L^2/\mathbb{R}} \lesssim \sup_v \frac{(p, \operatorname{div} v)}{||v||_1} \le ||p||_{L^2/\mathbb{R}}.$$

The lower bound follows from the inf-sup conditions on $L^2_0(\Omega) = L^2/\mathbb{R}$ and $V = H^1_0(\Omega)$. For the upper bound: given a constant $\alpha \in \mathbb{R}$,

$$\frac{(p, \operatorname{div} v)}{||v||_1} = \frac{(p+\alpha, \operatorname{div} v)}{||v||_1} \le ||p+\alpha||_0$$
$$\implies \sup_v \frac{(p, \operatorname{div} v)}{||v||_1} \le \inf_\alpha ||p+\alpha||_0 = ||p||_{L^2/\mathbb{R}}.$$

Therefore, we have uniform well-posedness of B^ε in the norm $X = Q \times V$, with $Q = L^2_0(\Omega)$ with norm

$$||p||^2_Q = ||p||^2_{L^2/\mathbb{R}} + \varepsilon||p||^2_0.$$

2.2.2 Mixed boundary conditions

If we have mixed boundary conditions (Dirichlet + Neumann), the inf-sup conditions hold over $Q = L^2(\Omega)$ (check) and we have

$$||p||_0 \lesssim \sup_v \frac{(p, \operatorname{div} v)}{||v||_1} \leq ||p||_0$$

so $||p||_0 = ||p||_{Q,0}$. This also implies that

$$||p||_0^2 \leq ||p||_{Q,0}^2 + \varepsilon ||p||_0^2 \leq 2||p||_0^2 \quad \forall \varepsilon \in (0,1]$$

and therefore we have uniform well-posedness of B^ε on $X = V \times Q$, with $Q = L^2(\Omega)$ with its usual norm.

2.3 The discrete system

The $\mathbb{P}_2 - \mathbb{P}_0$ elements are used for the discrete velocity and pressure spaces $X_h = V_h \times Q_h$. Then, the discrete bilinear form $A_h^\varepsilon : V_h \times V_h \to \mathbb{R}$ is given by

$$A_h^\varepsilon(u_h, v_h) = (e(u_h), e(v_h))_0 + \varepsilon^{-1}(\overline{\operatorname{div} u_h}^h, \overline{\operatorname{div} v_h}^h)_0.$$

2.3.1 Stability in X_h

We can prove that the inf-sup conditions hold in X_h by constructing an appropriate Fortin operator. This is carried out in [1, Section 8.4.3].

Definition 1 (Fortin operator)**.** *We call $I_h : V \to V_h$ a Fortin operator if*

1. *It is uniformly bounded, i.e.*

$$||I_h v|| \lesssim ||v|| \quad \forall v \in V.$$

2. *It satisfies*

$$c(\Lambda I_h v, q_h) = c(\Lambda v, q_h) \quad \forall v \in V, \forall q_h \in Q_h.$$

For our case, property 2 can be rewritten as follows:

$$\begin{aligned}
& (\operatorname{div}(I_h v), q_h)_0 = (\operatorname{div} v, q_h)_0 && \forall v, q_h \\
\iff & \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}(I_h v) q_h \, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}(v) q_h \, \mathrm{d}x && \forall v, q_h \\
\iff & \int_K \operatorname{div}(I_h v - v) \, \mathrm{d}x = 0 && \forall v, K \\
\iff & \int_{\partial K} (I_h v - v) \cdot n \, \mathrm{d}s = 0 && \forall v, K.
\end{aligned}$$

A standard way to construct such an operator (see [1, Proposition 5.4.4]) is by means of two operators I_h^1, I_h^2 from V onto V_h as follows:

Lemma 2. *Let I_h^1, I_h^2 be two operators from V onto V_h with the following properties:*

1. *I_h^1 is uniformly bounded,*
2. *I_h^2 satisfies*
$$c(\Lambda I_h^2 v, q_h) = c(\Lambda v, q_h)$$
(although in general it won't be uniformly bounded),
3. *$I_h^2(I - I_h^1)$ is uniformly bounded.*

Then
$$I_h^F = I_h^1 + I_h^2(I - I_h^1)$$
is a Fortin operator.

For the $\mathbb{P}_2 - \mathbb{P}_0$ elements, we set $I_h^1 : V \to V_h$ to be the standard Clement interpolator, which has the optimal approximation property

$$\sum_K h_K^{2r-2} |v - I_h^1 v|_{r,K}^2 \lesssim ||v||_1^2 \quad r = 0, 1.$$

As a result, setting $r = 1$, using the triangle and Poincaré inequality, we see that I_h^1 is uniformly bounded:

$$||I_h^1 v||_1 \leq ||I_h^1 v - v||_1 + ||v||_1 \lesssim ||v||_1.$$

We define $I_h^2 : V \to V_h$ by enforcing 6 (linearly independent) conditions for each component of the vector $I_h^2 v$ per triangle:

$$I_h^2 v|_K(M) = 0 \qquad \forall \text{M vertex of } K,$$
$$\int_e I_h^2 v \, \mathrm{d}s = \int_e v \, \mathrm{d}s \qquad \forall \text{e edge of } K.$$

This last condition implies that

$$\int_{\partial\Omega} (I_h^2 v - v) \cdot n \, \mathrm{d}s = 0$$

and therefore condition 2 holds for I_h^2. We now need to check condition 3. We use a scaling argument to show that

$$|I_h^2 v|_{1,K} = |\widehat{I_h^2 v}|_{1,\hat{K}} \leq c||\hat{v}||_{1,\hat{K}} \leq c \left(h_K^{-1} ||v||_{0,K} + |v|_{1,K} \right),$$

for $v \in V$ and $\hat{K} = h_K^{-1} K$ (so it has diameter 1 and the constants that arise in inequalities do not depend on h, only on minimum angles etc.). Now,

$$||I_h^2(I - I_h^1)u||_{1,K}^2 \leq c \left(h_K^{-2} ||(I - I_h^1)u||_{0,K} + ||(I - I_h^1)u||_{1,K}^2 \right) \leq ||u||_{1,K}^2.$$

3 Reissner-Mindlin plate

The first part of section 2.4.3 introduces the (primal) **scaled problem** and its dual counterpart. It also presents a regularity result, which motivates the introduction of the spaces V^+, V^- and Q^+ and the characterisation of the dual of V^-.

3.1 The discrete stabilised bilinear form

We work directly with the discrete problem because a stabilised h dependent bilinear form is considered. A **stabilisation parameter** $\mu \in (0, 1]$ is introduced. The **material parameter** is

$$\varepsilon = \frac{t^2}{k}.$$

3.1.1 Primal formulation

We have the primal space

$$V = H_0^1(\Omega) \times [H_0^1(\Omega)]^2$$

which represents the space of transverse displacements (scalar) and rotations (vector). The space V_h is $\mathbb{P}_2 \times [\mathbb{P}_1^+]^2$. **Stabilised primal formulation**:

$$A_h^\varepsilon((w_h, \beta_h), (v_h, \delta_h)) = a((w_h, \beta_h), (v_h, \delta_h)) + \varepsilon^{-1} c(\Lambda_h(w_h, \beta_h), \Lambda_h(v_h, \delta_h)),$$

where

$$\begin{aligned} a((w_h, \beta_h), (v_h, \delta_h)) &= a^b(\beta_h, \delta_h) + \frac{k\mu}{h^2 + t^2} (\nabla w_h - \beta_h, \nabla v_h - \delta_h)_0, \\ \Lambda_h(w_h, \beta_h) &= \overline{\nabla w_h - \beta_h}^h, \\ c(p_h, q_h) &= \left(1 - \frac{t^2 \mu}{h^2 + t^2}\right) ||p||_0^2. \end{aligned}$$

Here, $a^b(\cdot, \cdot)$ is uniformly continuous and coercive. Note that the bilinear forms depend on h and t. In order to have uniform continuity and coercivity for $a(\cdot, \cdot)$, we define the norm

$$||(w, \beta)||_V^2 := ||\beta||_1^2 + \frac{1}{h^2 + t^2} ||\nabla w - \beta||_0^2.$$

Regarding $|| \cdot ||_c$, uniform norm equivalence with $|| \cdot ||_0$ depends on the value of $\mu \in (0, 1]$:

$$\begin{aligned} &\text{if } \mu < 1: \quad (1 - \mu)||p||_0^2 \le ||p||_c^2 \le ||p||_0^2 \quad \Longrightarrow ||p||_0 \simeq ||p||_c, \\ &\text{if } \mu = 1: \quad ||p||_c^2 \simeq \min(1, h^2/t^2)||p||_0^2. \end{aligned}$$

For the last equivalence, note that if $\mu = 1$ then we have

$$1 - \frac{t^2}{t^2 + h^2} = 1 - \frac{1}{h^2/t^2 + 1} = \frac{h^2/t^2}{1 + h^2/t^2}.$$

Now, we consider the function $x/(1+x)$ for $x \geq 0$. We have

$$\frac{x}{1+x} \leq x \quad \text{and} \quad \frac{x}{1+x} \leq 1 \implies \frac{x}{1+x} \leq \min(1, x).$$

Also,

$$\begin{aligned} \text{if } 0 \leq x \leq 1: &\quad x(1+x) \leq 2x \implies \frac{1}{2}x \leq \frac{x}{1+x}, \\ \text{if } 1 \leq x: &\quad \frac{x}{1+x} \geq \frac{1}{2}, \end{aligned}$$

so

$$\frac{x}{1+x} \geq \frac{1}{2}\min(1, x).$$

3.1.2 Dual formulation

We introduce the dual variable

$$p = \varepsilon^{-1}\Lambda(w, \beta) = \frac{k}{t^2}(\nabla w - \beta)$$

which represents the scaled shear force. We have

$$Q = L^2(\Omega) \quad \text{and} \quad Q_h = \mathbb{P}_0.$$

We define the norm on Q such that we directly obtain stability:

$$||p||_Q^2 := \underbrace{\sup_{(w,\beta) \in V} \frac{c(\nabla w - \beta), p)^2}{||(w, \beta)||_V^2}}_{=: ||p||_{Q_0}^2} + \varepsilon ||p||_c^2.$$

Using the definition of $|| \cdot ||_V$ and Cauchy-Schwartz we get the upper bound

$$||p||_{Q_0} = \sup_{(w,\beta) \in V} \frac{c(\nabla w - \beta), p)^2}{||(w, \beta)||_V^2} \leq (h + t)||q||_c.$$

3.2 Stability of the discrete formulation

We construct a Fortin operator to prove that the discretisation

$$V_h = \mathbb{P}_2 \times [\mathbb{P}_1^+]^2 \quad Q_h = [\mathbb{P}_0]^2$$

is stable with respect to the operator

$$((w_h, \beta_h), q_h) \mapsto c(\nabla w_h - \beta_h, q_h).$$

We will show that the discrete inf-sup conditions hold by using **the relaxed criterion of Fortin.** We break Ω into two subsets

$$\Omega = \Omega_{h\leq t} \cup \Omega_{h>t}$$

and define

$$\begin{aligned} Q_{h\leq t} &= \{q \in Q_h : \text{supp}(q) \subset \Omega_{h\leq t}\} \\ Q_{h>t} &= \{q \in Q_h : \text{supp}(q) \subset \Omega_{h>t}\}\,,. \end{aligned}$$

We have

$$Q_h = Q_{h\leq t} \oplus Q_{h>t}$$

and

$$p_h = \underbrace{p_0}_{\in Q_{h\leq t}} + \underbrace{p_1}_{\in Q_{h>t}}, \quad ||p_h||_0^2 = ||p_0||_0^2 + ||p_1||_0^2.$$

Also, given the estimate for $||\cdot||_{Q_0}$, for $q_0 \in Q_{h\leq t}$,

$$||q_0||_Q \simeq ||q_0||_{Q_0} + \varepsilon^{1/2}||q_0||_c \leq (h + t + \varepsilon^{1/2})||q_0||_c \lesssim \varepsilon^{1/2}||q_0||_c.$$

So, in order to show stability, we have to find an operator

$$I_h^F : V_h \to V$$

that is **uniformly bounded** and

$$(1) \quad (w_h, \beta_h) = I_h^F((w,\beta)); \quad c(\nabla w_h - \beta_h, q_1) = c(\nabla w - \beta, q_1) \quad \forall q_1 \in Q_{h>t}.$$

Lemma 3. *Define the operator*

$$I_h^F((w,\beta)) = (0, \beta_h),$$

where

$$\beta_h = \sum_{T\in\mathcal{T}_h : h_T > t} ((\partial_1 w - \beta_1, 1)_T, (\partial_2 w - \beta_2, 1)_T)\,\frac{b_T}{(b_T, 1)_T},$$

b_T is the bubble function in T and $(\cdot,\cdot)_T$ is the L^2 inner product restricted to T. This operator is uniformly bounded and satisfies (1).

Proof. For uniform boundedness:

$$\begin{aligned} ||(0,\beta_h)||_V &\lesssim ||\beta_h||_1 + \frac{1}{h+t}||\beta_h||_0 \\ &\lesssim h^{-1}||\beta_h||_0 && (h > t \text{ and } ||\nabla b_T||_0 \lesssim (h+t)^{-1}||b_t||_0) \\ &\lesssim (h+t)^{-1}||\nabla w - \beta||_0 && (\text{Hölder's inequality}) \\ &\leq ||(w,\beta)||_V. \end{aligned}$$

Regarding the invariance property (1), its a straightforward consequence of the definition:

$$\begin{aligned} (\nabla w_h - \beta_h, q_1) &= \sum_{T\in\mathcal{T}_h : h_T > t} q_1|_T(\partial_1 w - \beta_1, 1)_T + q_2|_T(\partial_2 w - \beta_2, 1)_T \\ &= (\nabla w - \beta, q_1). \end{aligned}$$

□

3.3 A priori estimates

A result of having a stable discretisation is that the solution of the discrete mixed problem is a **best approximation**, in the sense that

$$
\begin{aligned}
&||(w,\beta)-(w_h,\beta_h)||_V+||p-p_h||_Q \\
&\lesssim \inf_{(v_h,\eta_h,q_h)\in X_h}\left(||(w,\beta)-(v_h,\eta_h)||_V+||p-q_h||_Q\right).
\end{aligned} \tag{2}
$$

We also have the **regularity result**:

$$
||(w,\beta)||_{V^+}+||p||_{Q^+}\lesssim ||(g,\delta)||_{(V^-)^*}, \tag{3}
$$

where

$$
\begin{aligned}
||(w,\beta)||_{V^+}^2 &= \inf_{w=w_0+w_r}\left(||w_0||_3^2+t^{-2}||w_r||_2^2\right)+||\beta||_2^2,\\
||(w,\beta)||_{V^-}^2 &= \inf_{w=w_0+w_r}\left(||w_0||_1^2+t^{-2}||w_r||_0^2\right)+||\beta||_0^2,\\
||q||_{Q^+}^2 &= ||p||_0^2+t^2||p||_1^2.
\end{aligned}
$$

We also have **optimal interpolation operators in $H^k(\Omega)$-norms**

$$
\begin{aligned}
&I_h^V=(I_h^{V,w},I_h^{V,\beta}):V\to V_h,\\
&I_h^Q:Q\to Q_h
\end{aligned}
$$

such that

$$
||w-I_h^{V,w}w||_k\lesssim h^{l-k}||w||_l \qquad 0\le k\le 1, k\le l\le 3, \tag{4}
$$

$$
||\beta-I_h^{V,\beta}\beta||_k\lesssim h^{l-k}||\beta||_l \qquad 0\le k\le 1, k\le l\le 2, \tag{5}
$$

$$
||q-I_h^{Q}q||_k\lesssim h^{l-k}||q||_l \qquad 0\le k\le l\le 1. \tag{6}
$$

Our goal is to **obtain approximation estimates in $V\times Q$ norms**. Once we have these, the a priori estimates follow from (2) and (3).

3.3.1 Some technical lemmas

Lemmas 2.17 and 2.18 are used to characterise the norm $||\cdot||_{V^{-,w}}$. In order to do so, a local regularisation operator at length scale t

$$
I_t^w:V^{-,w}\to V
$$

with certain approximation properties is introduced and assumed to exist.

3.3.2 Approximation estimates in $V\times Q$ norms

Theorem 4. *The interpolation operators have full order of approximation in the $V\times Q$ norms:*

$$
h^{-1}||(w,\beta)-I_h^V(w,\beta)||_{V^-}+||(w,\beta)-I_h^V(w,\beta)||_V\lesssim h||(w,\beta)||_{V^+}, \tag{7}
$$

$$
h^{-1}||(w,\beta)-I_h^V(w,\beta)||_{V^-}+||I_h^V(w,\beta)||_V\lesssim ||(w,\beta)||_V, \tag{8}
$$

$$
||I_h^V(w,\beta)||_{V^-}\lesssim ||(w,\beta)||_{V^-}, \tag{9}
$$

and

$$\|p - I_h^Q p\|_Q \lesssim h\|p\|_{Q^+}. \tag{10}$$

Proof. **(V^- norm in (7).)** We have

$$\|(w,\beta) - I_h^V(w,\beta)\|_{V^-} = \inf_{w=w_0+w_r} \left(\|w_0\|_1^2 + t^{-2}\|w_r\|_0^2\right) + \|\beta\|_0^2.$$

For $\|\beta\|_0$ we just use the property of the $I_h^{V,\beta}$, that is, (6). For the w component, we use an adequate decomposition. Since

$$(a^2+b^2)^{1/2} \geq a^2 + b^2 \quad \forall a, b \geq 0,$$

for any $w \in V^w$ we can find a decomposition $w = w_0 + w_r$ such that

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 \leq \|w\|_{V^{+,w}}.$$

Then using this inequality and the interpolation properties of $I_h^{V,w}$, that is, (4), we get

$$\|w - I_h^{V,w} w\|_{V^{-,w}} \lesssim h^2 \|w\|_{V^{+,w}}.$$

(V norm in (7).)

$$\begin{aligned}
&\|(w,\beta) - I_h^V(w,\beta)\|_V \\
&\lesssim \|\beta - I_h^{V,\beta}\beta\|_1 + (h+t)^{-1}\|\nabla(w - I_h^{V,w}w) - (\beta - I_h^{V,\beta}\beta)\|_0 \\
&\lesssim h\|\beta\|_2 + (h+t)^{-1}\|(w_0+w_r) - I_h^{V,w}(w_0+w_r)\|_1 \\
&\lesssim h\|\beta\|_2 + (h+t)^{-1}\left(h^2\|w_0\|_3 + h\|w_r\|_2\right) \\
&\lesssim h\|\beta\|_2 + h\left(\|w_0\|_3 + t^{-1}\|w_r\|_2\right)
\end{aligned}$$

so taking the infimum we have the result we want.
(V^- norm in (8).) We want to show that

$$\|(w,\beta) - I_h^V(w,\beta)\|_{V^-} \lesssim h\|(w,\beta)\|_V.$$

To do so, we shall use the regularisation operator from Lemma 2.17. We concentrate on the $V^{-,w}$ component. We use the splitting

$$w - I_h^{V,w}w = I_t^w\left(w - I_h^{V,w}w\right) + (I - I_t^w)\left(w - I_h^{V,w}w\right)$$

and the definition of the V^- norm:

$$\begin{aligned}
\|w - I_h^{V,w}w\|_{V^{-,w}} &\lesssim \|I_t^w\left(w - I_h^{V,w}w\right)\|_1 + t^{-1}\|(I - I_t^w)\left(w - I_h^{V,w}w\right)\|_0 \\
&= \|I_t^w\left(w - I_h^{V,w}w\right)\|_{1,\Omega_{h>t}} + t^{-1}\|(I - I_t^w)\left(w - I_h^{V,w}w\right)\|_{0,,\Omega_{h>t}} \\
&\quad + \|I_t^w\left(w - I_h^{V,w}w\right)\|_{1,\Omega_{h\leq t}} + t^{-1}\|(I - I_t^w)\left(w - I_h^{V,w}w\right)\|_{0,,\Omega_{h\leq t}}
\end{aligned}$$

Then, we use the properties of the regularisation operator (approximation on $\Omega_{h>t}$ and uniform coninuity in $||\cdot||_{\Omega_{h\leq t}}$), see Lemma 2.17, to obtain

$$||w - I_h^{V,w}w||_{V^{-,w}} \lesssim ||w - I_h^{V,w}w||_{1,\Omega_{h>t}} + t^{-1}||w - I_h^{V,w}w||_{0,\Omega h\leq t}.$$

Now, by a scaling argument (I think...),

$$||w - I_h^{V,w}w||_{0,\Omega h\leq t} \leq h||w - I_h^{V,w}w||_{1,\Omega h\leq t}$$

so that

$$\begin{aligned}||w - I_h^{V,w}w||_{V^{-,w}} &\lesssim ||w - I_h^{V,w}w||_{1,\Omega_{h>t}} + ht^{-1}||w - I_h^{V,w}w||_{1,\Omega_{h\leq t}}\\ &\lesssim \frac{h}{h+t}||w - I_h^{V,w}w||_1\end{aligned}$$

where we have used:

$$\begin{aligned}h+t<2h &\implies 1 \lesssim \frac{h}{h+t} \quad \text{on } \Omega_{h>t},\\ h+t\leq 2t &\implies ht^{-1} \lesssim \frac{h}{h+t} \quad \text{on } \Omega_{h\leq t}.\end{aligned}$$

See the thesis for the last step.
(V norm in (8); uniform continuity in V.) Since

$$||(w,\beta)||_V^2 := ||\beta||_1^2 + \frac{1}{h^2+t^2}||\nabla w - \beta||_0^2.$$

and we already have uniform continuity of $I_h^{V,\beta}$ in the $||\cdot||_1$ norm, we need to do something about the second part of the V-norm. We will show that I_h^V is uniformly continuous in that norm by first showing that it is uniformly continuous in the following seminorm:

$$|(w,\beta)|_V^2 := ||\nabla\beta||_0^2 + ||\nabla w - \beta||_0^2.$$

Schöberl shows this by using the Bramble-Hilbert lemma. This essentially boils down to the following: Note that

$$V_{00} = \left\{(w,\beta) = (a + b^T x, b) : a \in \mathbb{R}, \quad b \in \mathbb{R}^2\right\} = \mathrm{Ker}\,(|\cdot|_V)$$

and

$$V_{00} \subset \mathrm{Ker}\left(I - I_h^V\right) = V_h.$$

Then, by a standard isomorphism theorem on normed vector spacces, $|\cdot|_V$ is a norm on the quotient space V/V_{00} equivalent to its canonical norm, i.e.

$$|(w,\beta)|_V \simeq \inf_{(v,\eta)\in V_{00}} ||(w,\beta) - (v,\eta)||_1 \tag{11}$$

and, using the fact that I_h^V is continuous in $||\cdot||_1$ and $I - I_h^V$ is surjective onto V, we have that $I - I_h^V$ is an isomorphism from V/V_h onto V, i.e.

$$||(w,\beta) - I_h^V(w,\beta)||_1 \simeq \inf_{(v,\eta)\in V_h} ||(w,\beta)-(v,\eta)||_1. \tag{12}$$

Also, on the reference element, we have the following norm equivalence:

$$||(w,\beta)||_1 \simeq ||(w,\beta)||_0 + |(w,\beta)|_V. \tag{13}$$

Therefore, on the reference element (so we ignore h's that may arise in the $\lesssim$), we can deduce that

$$\begin{aligned}
|(w,\beta) - I_h^V(w,\beta)|_V &\lesssim ||(w,\beta) - I_h^V(w,\beta)||_1 && \text{(by (13))}\\
&\lesssim \inf_{(v,\eta)\in V_h} ||(w,\beta)-(v,\eta)||_1 && \text{(by (12))}\\
&\leq \inf_{(v,\eta)\in V_{00}} ||(w,\beta)-(v,\eta)||_1 && (\ V_{00}\subset V_h)\\
&\lesssim |(w,\beta)|_V. && \text{(by (11))}
\end{aligned}$$

Using the triangle inequality, we see that

$$|I_h^V(w,\beta)|_V \lesssim |(w,\beta)|_V$$

holds on the reference element. By a scaling argument, we have continuity with respect to the seminorm

$$||\nabla\beta||_0^2 + h^{-2}||\nabla w - \beta||_0^2.$$

By the properties of the interpolation operator I_h^V, we also have continuity with respect to the seminorm

$$||\nabla\beta||_0^2$$

so therefore the operator is uniformly continuous with respect to the family of seminorms

$$||\nabla\beta||_0^2 + \alpha||\nabla w - \beta||_0^2 \quad \forall\alpha \lesssim h^{-2}$$

and in particular, this holds for $\alpha = (h^2+t^2)^{-1}$ and this establishes the uniform continuity of the operator with respect to the V-norm.
(V^- norm in (9); uniform continuity in V^-.)
(Q norm in (10))

□

3.3.3 An inverse estimate

References

[1] D. Boffi, F. Brezzi, and M. Fortin, *Mixed finite element methods and applications*, Springer Series in Computational Mathematics, Springer, 2013.

[2] Patrick E. Farrell, Lawrence. Mitchell, and Florian. Wechsung, *An Augmented Lagrangian Preconditioner for the 3D Stationary Incompressible Navier–Stokes Equations at High Reynolds Number*, SIAM Journal on Scientific Computing **41** (2019), no. 5, A3073–A3096.

[3] Qingguo Hong, Johannes Kraus, Jinchao Xu, and Ludmil Zikatanov, *A robust multigrid method for discontinuous Galerkin discretizations of Stokes and linear elasticity equations*, Numerische Mathematik **132** (2015).